The Classification of Plane Involutions of Order (3).

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Introduction and General Discussion.

1. The purpose of this paper is to discuss all the different algebraic (1, 3) point correspondences between two planes. Such a correspondence is established between two planes (x) and (y) when a relation exists such that to any point in (x) corresponds a point (y), but to any point in (y) corresponds three points (x).

The images of lines on either plane are curves of order m in the other. The simple plane (x) contains curves which are nets, images of lines in (y).

These curves have n variable intersections which are the n points of (x) corresponding to a point (y) determined by the intersection of two lines whose images are the two curves; the remaining $m-n$ intersections are fixed and are common to all the image curves of the net. If two of the n points coincide, two curves of the net are tangent at the point. The locus of all these points of tangency is the curve of coincidences. We shall designate it by K. It is always a component of the Jacobian of the net.

There exists in the (y) plane a curve of branch points whose points are in (1, 1) correspondence with those of K. This curve of branch points, denoted by L, is a fixed curve, every point of which has at least two of its n image points coinciding in a definite direction at a point on the coincidence curve. In case $m>2$ the remaining non-coincident image points describe another locus, the residual image of L. This locus will be denoted by J'. The points at which more than two of the n images of a point in (y) coincide are intersections of the residual image with the curve of coincidences.

The first (1, 2) transformations discussed were those by Geiser.*

In this discussion a cubic surface is projected on a double plane from a point on it, and also mapped on a single plane by means of two non-intersecting lines on the surface.

In 1871 Clebsch* devised a (1, 2) point correspondence between two planes depending on the bisection of Abelian functions connected with plane

* "Ueber zwei geometresche Probleme," *Crelle's Journal für die Mathematik,* Vol. LXVII (1867), pp. 78–89.

conics, the plane of the conic being so related to a surface that to a point on the surface corresponds one point in the plane, but to a point in the plane corresponds two points on the surface.

The next step was made by de Paolis † who worked out the general theory of (1, 2) point correspondence between two planes as a generalization of the rational transformation of Cremona.

This is the first published treatment of multiple correspondence by means of an algebraic method.

The classification of all possible birational (involutorial) transformations associated with the (1, 2) types was completed by Bertini. ‡

Also F. M. Morgan. § In 1884 Chizzoni ‖ developed the general theory of (1, n) point correspondence by a plane involution of order n in which he considered a given plane as containing ∞^2 series of groups of n points such that each group is fully determined by any one of the points.

Other writers that have made noteworthy contributions to the theory of (1, n) point correspondence between two planes are G. Castelnuovo,¶ Miss Charlotte A. Scott,** and Amerigo Bottari.††

All of these treat the subject for a general n except Miss Scott who, in addition, works out several examples for special values of n greater than 2.

Enumeration of Independent Types.

2. In the theory of (1, 3) point correspondence between two planes there are five types, no one of which can be derived from another by means of a

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* "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen," *Mathematische Annalen*, Vol. III (1871), pp. 45–75.

† (1) "Le transformazioni doppia."

(2) "La transformazione piano doppia di secondo ordine, e le suo applicazione alla geometria non euclidea."

(3) "La transformazione piano doppia di terzo ordine prime genere, e la sua applicazione a curve del quarto ordine," *Atti della R. Accademia dei Lincei*, Series 3, Vol. XII (1877).

‡ "Ricerche sulle transformazioni univoche involutorie nel piano," *Annali di Matematica*, Series 2, Vol. VIII (1877), pp. 244–286.

§ "Involutorial Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), pp. 79–104.

‖ "Sopra le involuzioni piani," *Atti della R. Accademia dei Lencei*, Series 3, T. 19 (1884), pp. 301–371.

¶ "Sulla razionalita delle involuzioni piane," *Mathematische Annalen*, Vol. XLIV (1894), pp. 125–155.

** "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. XXIX (1899), pp. 329–381, and *ibid.*, Vol. XXXII (1901), pp. 209–239.

†† "Sulla razionaliti dei piani multipli $\{xy\sqrt[n]{F(x,y)}\}$," *Giornale di Matematiche*, Vol. XLI (1903), pp. 285–320, and also "Sulla razionaliti dei piani multipli $\{x, y\sqrt[n]{F(x,y)}\}$," *Annali di Matematica*, Series 3, Vol. II (1899), pp. 277–296.

birational transformation. For each type there exists in one plane a point determined by two straight lines, and in the other three image points. The five distinct types are defined in the following ways:

(1) a. A field of straight lines, $\Sigma y_i x_i = 0$.

b. A net of cubic curves $\phi_i = 0$ without basis points, $\Sigma y_i \phi_i = 0$.

(2) a. A pencil of lines, $y_1 x_1 + y_2 x_2 = 0$.

b. A net of curves of order higher than 3 and having a fixed point of multiplicity $n-3$ at the vertex of the pencil of lines $y_1 x_1 + y_2 x_2 = 0$. This is associated with Jonquières' configuration in (1, 1) and (1, 2) transformations. A discussion of the (1, 1) type has been presented by P. P. Boyd.*

(3) a. A net of conics through one fixed point.

b. A net of conics through the same fixed point,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ are general conics through a fixed point.

(4) a. A net of cubic curves passing through six fixed points.

b. A net of cubic curves passing through the same fixed points,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ now represent cubic curves through six basis points.

(5) a. A pencil of cubic curves through nine fixed points.

b. A net of curves of order 9 passing triply through eight of the nine fixed points,

$$y_1 \psi_1 + y_2 \phi_2 = 0, \quad \Sigma y_i \psi_i = 0.$$

If u_i is of order m_1 and v_i is of order m_2, the transformation curves are of order $m_1 + m_2 = m$. To the line $y_i = 0$ correspond curves $\phi_i(x) = 0$ of order m; to the lines $x_i = 0$ correspond curves $\psi_i(y) = 0$ of order m.

The fixed points common to all curves of the defining system $u_i = 0$, $v_i = 0$ are fundamental. To each fundamental point F corresponds a curve $f(y)$ whose order is equal to the multiplicity of the point on the curves $\phi_i(x) = 0$, and to each direction through the fundamental multiple point corresponds a point on the image fundamental curve $f(y)$. The complete image of $f(y)$ is composite. The components are the original multiple point and a residual curve.

Every branch of this residual curve has contact with some branch of K through this point. To each of these simple intersections on L is related a

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* "On the Perspective Jonquières Involutions Associated with the (2, 1) Ternary Correspondence." AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 290–324.

direction of J' through the original point. Aside from these intersections J' has only points of contact in common with the residual curve and these correspond to the points of tangency of L with $f(y)$. See F. Chizzoni.*

If two curves of the defining system have contact at a given point, then one curve of the net has a double point at the point. Hence the locus of all the double points of curves of the net is the curve K.

The order of the Jacobian is $3(m-1)$. The multiplicity of any fundamental point P_i on the $\phi_i=0$ which lies on the Jacobian is $3i-1$.

The equation of L may be obtained by eliminating (x_i) from the equation for K and the equations of transformation $y_i=\phi_i(x)$.

The order of the curve L is the number of intersections aside from those at the fundamental points of K with the image of a straight line (y). This has been expressed by Zeuthen † in the formula $2(k+p-1)$ where k is the multiplicity of the transformation and p is the genus of the curves of transformation, images of the straight lines in (y).

The complete image in (x) of L is a composite curve which has for its order the product of the orders of L and the transformation ϕ. The image contains K twice and the residual curve J', and also contains fundamental curves, when L passes through their corresponding fundamental points.

Type I.

3. The defining equations for this type are

$$x_1y_1+x_2y_2+x_3y_3=0, \quad u_1y_1+u_2y_2+u_3y_3=0,$$

where $u_i=0$ are cubic curves having no fixed point in common. On solving for y_i we obtain

$$\rho y_1=u_3x_2-u_2x_3=\phi_1(x), \quad \rho y_2=u_3x_1-u_1x_3=\phi_2(x), \quad \rho y_3=u_1x_2-u_2x_1=\phi_3(x).$$

To obtain the points common to the three curves $\phi_i=0$ we write the equations in the form $\frac{u_2}{x_2}=\frac{u_3}{x_3}=\frac{u_1}{x_1}$. Of the sixteen intersections of any two curves as $\phi=0$, $\phi_3=0$ we exclude those which make $u_1=0$, $x_1=0$ as these points do not lie on $\phi_1=0$. Hence we find $16-3=13$ points common to all the curves of the net. These are simple fundamental points.

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* "Sopra le involuzioni piane," *Lincei Memoire,* Series 3, Vol. XIX (1884), p. 301.

† "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes," *Mathematische Annalen,* Vol. III (1871), p. 150.

The number of conditions required to determine a quartic curve is fourteen. Since there are thirteen fixed points on each quartic curve, any one of the three variable points uniquely fixes the other two.

The genus of the curves $\phi_i=0$ is 3 since they have no double points. The image curve is a rational quartic curve and has a triple point at (a_1, a_2, a_3). Two of these quartic curves intersect in sixteen points, each of which has three image points in (x). The complete image of this quartic curve is a curve of order 16. This curve is composed of the original line and a curve of order 15. Two of these curves intersect in two hundred and twenty-five points. But if each complete curve passes through each of the thirteen fundamental points four times, aside from these intersections there are $225-4 \cdot 4 \cdot 13=17$ points. But the two straight lines intersect in one image point, and each line cuts the curve of order 15 in fifteen points. The forty-eight image points are made up of seventeen points on the two curves of order 15, not at the thirteen fundamental points; one point of intersection of the two straight lines and fifteen points on each line through the curve of order 15 with which it is related.

The curve K is of order 9 with double points at the thirteen basis points. Hence it is of genus 15. The number of variable intersections of K with the image in (x) of a straight line in (y) determines the order of the image of K. The curve L is, then, of order 10. $9 \cdot 4-2 \cdot 13=10$. It is also of genus 15, hence it has twenty-one double points or their equivalents. The curve J' is of the order 22 and genus 15, hence has one hundred and ninety-five double points or their equivalents.

The number of intersections of K with J' aside from those at the fixed points are forty-two, or twenty-one contacts, $9 \cdot 22-13 \cdot 2 \cdot 6$. But this is the number of double points of L to be accounted for.

It will be shown that the forty-two intersections of K with J' are twenty-one contacts, images of cusps on L. The equation of L is the condition that a line in (x) touches its associated cubic, hence it is the discriminant of a binary cubic equated to zero. These values that make the binary cubic in x_2, x_3 a perfect cube correspond to cusps on L since the form of the discriminant is $4G^2+H^3$. Any point on L that makes $H=0$ and $G=0$ is such that all three images of the point are coincident. Hence in (x) K and J' have the image point in common.

The cuspidal tangent to L corresponds to the particular quartic of the net which has a node at the corresponding point on K.

Fundamental System.

4. Each of the thirteen points of the net of quartics when substituted in the defining equations, makes the two equations identical, hence they are all fundamental points, and their images are straight lines in (y). There are no fundamental points in (y), hence no fundamental curves in (x).

The complete image of each of these lines is a curve in (x) of order 4 which has a double point at the original point. For, let $\bar{x}_1 y_1 + \bar{x}_2 y_2 + \bar{x}_3 y_3 = 0$ be the equation of the fundamental line in (y) which is the image of the fundamental point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ which is any of the thirteen points (basis) of the net of quartics.

The image of this in (x) is obtained by making the transformation according to the equations already obtained and we have

$$\bar{x}_1 (u_3 x_2 - u_2 x_3) + \bar{x}_2 (u_1 x_3 - u_3 x_1) + \bar{x}_3 (u_2 x_1 - u_1 x_2) = 0.$$

Differentiating this with regard to x_1 we have,

$$\bar{x}_1 \left(x_2 \frac{du_3}{dx_1} - x_3 \frac{du_2}{dx_1} \right) + \bar{x}_2 \left[x_3 \frac{du_1}{\cdot dx_1} - \left(u_3 + x_1 \frac{du_3}{dx_1} \right) \right] + \bar{x}_3 \left(u_2 + x_1 \frac{du_2}{dx_1} - x_2 \frac{du_1}{dx_1} \right) = 0,$$

substituting in this the coordinates of the point (x_1, x_2, x_3), we have the form written as the expression

$$\bar{x}_1 \left(\bar{x}_2 \frac{d\bar{u}}{dx_1} - x_3 \frac{d\bar{u}_2}{dx_1} \right) + x_2 \left(\bar{x}_3 \frac{d\bar{u}_1}{dx_1} \bar{u}_3 - \bar{x}_1 \frac{du_3}{dx_1} \right) + x_3 \left(\bar{u}_2 + x_1 \frac{d\bar{u}_2}{dx_1} - x_2 \frac{d\bar{u}_1}{dx_1} \right) = 0.$$

But the only terms which do not vanish are $\bar{u}_2 \bar{x}_3 - \bar{u}_3 \bar{x}_2 = 0$, which equation is satisfied since $(\bar{x})$ is on $u_2 x_3 - u_3 x_2 = 0$. This quartic curve has one double point and twelve simple fundamental points. The intersections of K and J' with this image curve aside from those at the fundamental points are eight $(9 \cdot 4 - 2 \cdot 2 - 2 \cdot 12 = 8)$ for K, and four for J' $(4 \cdot 22 - 2 \cdot 6 - 6 \cdot 12 = 4)$.

At the common double point K touches each branch of the quartic. This is shown as follows: When two of the image points coincide, two or more of the curves are tangent at the basis points of the net and one curve has a double point. If the Jacobian has a double point at a basis point, it is tangent to each branch of the nodal curve of the net at this point.

The Jacobian can then be formed from a net of curves, one of which has at a fundamental point $(\bar{x}_1) = (0, 0, 1)$, a double point. These curves are

$$\psi_1 = (x_1 x_2)^2 x_3^2 + (x_1 x_2)^3 x_3 + (x_1 x_2)^4 = 0,$$
$$\psi_2 = u_1 (x_1 x_2) x_3^3 + u_2 (x_1 x_2) x_3^2 + u_3 (x_1 x_2) x_3 + u_4 (x_1 x_2) = 0,$$
$$\psi_3 = v_1 (x_1 x_2) x_3^3 + v_2 (x_1 x_2) x_3^2 + v_3 (x_1 x_2) x_3 + v_4 (x_1 x_2) = 0.$$

The Jacobian is of the form

$$\left|\begin{array}{lll} x_1x_2^2x_3^2+- & x_2(x_1)_2x_3^2 \ +-- & 2(x_1x_2)_2x_3+(x_1x_2)x_3 \ +-- \\ u_1(x_1x_2)x_3^2+- & u_2(x_1x_2)x_3^2 \ +-- & 3u(x_1x_2)x_3+u_1(x_1x_2)_3+-- \\ v_1(x_1x_2)x_3^2+- & v_2(x_1x_2)^2x_3^2+-- & 3v(x_1x_2)_2x_3+v_1(x_1x_2)_3+-- \end{array}\right|.$$

By collecting the coefficients of x_3 the expanded form is found to be

$$3x_2u_2v-3x_2v_2u-3u_1x_1v+2u_1v_2x_1x_2+3x_1v_1u-2u_2v_1x_1x_2=0,$$

or

$$3v(x_2u_2-u_1x_1)-3u(x_2v_2-x_1v_1)+2x_1x_2(u_1v_2-u_2v_1)=0.$$

But

$$u=ax_1+bx_2, \quad \frac{du}{dx_1}=a=u_1, \quad \frac{du}{dx_2}=b=u_2,$$

and

$$v=cx_1+dx_2, \quad \frac{dv}{dx_1}=c=v_1, \quad \frac{dv}{dx_2}=d=v_2.$$

The Jacobian becomes

$$3(cx_1+dx_2)(bx_2-ax_1)-3(ax_1+bx_2)(dx_2-cx_1)+2x_1x_2(ad-bc)=0,$$

or

$$3[bcx_1x_2+bdx_2^2-acx_1^2-adx_1x_2-adx_1x_2-bdx_2^2+acx_1^2+bcx_1x_2 \\ +2x_1x_2(ad-bc)]=0,$$

or finally $$2x_1x_2(bc-ad)=0.$$

This is the coefficient of the highest power which does not vanish in the equations of the tangents to the curves of the net passing through (x_1) and is also the coefficient of highest power of x_3 in the equation of the Jacobian. It is the form of the tangent to the Jacobian also, and we have proved that the nodal ϕ has the same tangent as the Jacobian at the double point. These are $x_1=0$ and $x_2=0$.

Each direction of J' through the point corresponds to one point of intersection of L with the fundamental line. The four remaining points of intersection on J' are two contacts, the images of the two points of contact of L with the fundamental line.

The line in (y) is bitangent to L. The six simple intersections are images of the six directions on J' at the fundamental point.

Type II.

5. The defining equations for Type II are

$$x_1y_1+x_2y_2=0, \tag{1}$$

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{2}$$

where $u_i=0$ represent curves of order $n>3$ having an $n-3$-fold point in common. Let this point be $P(x)=(0,0,1)$. Then

$$\rho y_1=u_3x_2=\phi_1(x), \quad \rho y_2=-u_3x_1=\phi_2(x), \quad \rho y_3=u_2x_1-u_1x_2=\phi_3(x).$$

These curves $\phi_i=0$ have an $(n-2)$-fold point at $P(x)$. The equation of the image of a general line in (x) has the form

$$\text{(a)} \quad \begin{aligned} &y_1\{(a_1y_2-a_2y_1)^3[f_1(a_3-y_1y_{2_{n-3}})]+(a_1y_2-a_2y_1)^2[f_1(a_3-y_1y_{2_{n-2}})] \\ &+(a_1y_2-a_2y_1)[f_1(a_3-y_1y_{2_{n-1}})]+f_1(a_3-y_1y_{2_n})\} \\ &+y_2\{(---)\}+y_3\{(---)\}=0. \end{aligned}$$

Let

$$f_i(a_3,\ -y_1y_{2_{n-3}})=v_i, \qquad f_i(a_3,\ -y_1y_{2_{n-1}})=s_i,$$
$$f_i(a_3,\ -y_1y_{2_{n-2}})=w_i, \qquad f_i(a_3,\ -y_1y_{2_n})\ =t_i,$$

the equation of the curve is then

$$\text{(b)} \quad \begin{aligned} &(a_1y_2-a_2y_1)^3(y_1v_1+y_2v_2+y_3v_3)+(a_1y_2-a_2y_1)^2(y_1w_1+y_2w_2+y_3w_3) \\ &+(a_1y_2-a_2y_1)(y_1s_1+y_2s_2+y_3s_3)+(y_1t_1+y_2t_3+y_3t_3)=0. \end{aligned}$$

This curve is of order $n-3+3+1=n+1$ having at $P(y)$ an n-fold point.

Aside from the intersections at $P(y)$ two curves of order $n+1$ will have $(n+1)^2-(n)^2=2n+1$ points in common, of which one is variable.

The image of a line through $P(x)$ is found by dividing the above equation (a) by a_3^{n-3} and then putting $a_3=0$. All terms containing a_3 to a power greater than $n-3$ will vanish. The remaining terms have the form

$$(a_1y_2-a_2y_1)^3[y_1f(y_1y_{2_{n-3}})+y_2f_2(y_1y_{2_{n-3}})+y_3f_3(y_1y_{2_{n-3}})]=0,$$

which consists of a line $a_1y_2-a_2y_1=0$ through $P(y)$ taken three times, and a curve $\Sigma y_if_i(y_1y_{2_{n-3}})=0$ of order $n-2$, having at $P(y)$ an $n-3$-fold point, and is independent of the coefficients a_i. The line is the image of the given line, and the curve is the image of the point $P(x)$.

The Fundamental System.

6. The $6(n-1)$ basis points are all fundamental; their images in (y) are straight lines through $P(y)$.

The complete image in (x) of each of these fundamental lines in (y) is a curve of order $n+1$ having an $(n-2)$-fold point at $P(x)$. But, since the line in (y) passes through $P(y)$, the image of $P(y)$ must be deducted. It is a curve C_1 passing through $P(x)$, and the one point of the $6(n-1)$ basis points of which the line in (y) is an image. The proper image in (x) of the line in (y) is, then, of order n having an $n-3$-fold point at $P(x)$ and passing once through each of the $6(n-1)$ points.

The point $P(x)$ of order $n-2$ on the transformation $\phi_i=0$ has a curve of order $n-2$ as its image in (y), with a point of multiplicity $n-3$ at $P(y)$. The complete image of this curve in (x) is of order $(n+1)(n-2)-n(n-3)=2(n-1)$, having at $P(x)$ a multiplicity $2n-4$ and passing once through each of the $6(n-1)$ points.

The curve K is of order $3n$, having at $P(x)$ a point of multiplicity $n-2+n-2+n-3=2n-7$. But u_3 is a factor of each term, so that the curve K is of order $2n$, having a $2n-4$-fold point. It also passes simply through the $6(n-1)$ simple basis points. The genus of K is $6n-9$.

The order of L is $2n(n+1)-(n-2)(2n-4)-6n-6=4n-2$, and its genus is $6n-9$. The curve has at $P(y)$ a point of multiplicity $4n-6$ and a double point at each of the $6(n-1)$ basis points. The curves K, J' have $6(n-1)$ contacts and L has $6(n-1)$ cusps.

Let P_i denote a simple fundamental point in (x); p_i denote the image in (y) of P_i, and let r_i denote the residual image curve in (x) of p_i.

Thus J' has a double point at P_i and one contact not at the fundamental point corresponding to the tangency of L with p_i.

We denote the point in (x) by Q, its image curve in (y) by q and the residual curve in (x) by s.

K passes through Q $2n-4$ times and is tangent to each of the $n-2$ branches of ϕ_i. There are $2n-4$ points of tangency of L with q, leaving $4n-6$ simple intersections. These simple intersections correspond to the directions of J' through Q.

The intersections of K with s, not at the fundamental points are shown to be $4n-6$. These intersections correspond to the simple intersections of L with q. Aside from the fundamental points, J' has $4n-8$ intersections or $2n-4$ contacts with s which correspond to the points of contact of L with q.

Type III.

7. The defining equations for Type III are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$, $v_i=0$ are general conics through $P=(0,0,1)$.

The Fundamental System.

8. The image of a point $P(x)$ is a curve $p(y)$ of order 2 in (y). The complete image in (x) of $p(y)$ is a curve of order 8 having a five-fold point at $P(x)$ and double points at each of the nine simple basis points.

The image of each of the nine simple basis points in (y) is a line whose complete image in (x) is a curve of order 4 belonging to the net having a double point at $P(x)$ and also at the given basis point.

K is of order 9, genus 9. There are no fundamental points in the (y) plane. The complete image of K consists of the conic $p(y)$ taken five times; images of the double point $P(x)$ through which K passes five times; nine straight lines each counted twice, images of the nine simple basis points, through which K passes twice, and also the curve of branch points L.

L is of order 8, genus 9; it has twelve cusps. J' has a six-fold point at $P(x)$; J' and K have twelve points of contact.

We shall denote one of the nine basis points of the net by Q_i, the straight line in (y) which is its image by q_i and the residual image curve in (x) by s. We shall denote the image of the fundamental point $P(x)$ of multiplicity 2 on ϕ_i by p and its residual curve in (x) by r.

As in Type I, L is tangent to q as many times as K passes through Q. Since K has a double point at each of the nine simple basis points, q is tangent to L twice. L and q have $1 \cdot 8 - 2 \cdot 2 = 4$ simple intersections, each of which corresponds to a direction of J' through Q.

K has four simple intersections with s not at fundamental points, the images of the four simple intersections of L with q.

J' has four simple intersections or two contacts with s corresponding to the two contacts of L with q.

Corresponding to the five contacts of K with r at $P(x)$ are five contacts of L with p, which leaves six simple intersections of p with L. These correspond to the directions of J' through $P(x)$.

K has six simple intersections with r besides those at the fundamental points. These are images of the simple intersections of L with p.

J' has ten simple intersections or five contacts with r not at fundamental points. These correspond to the five contacts of L with p.

In the general case of two nets of conics through a common point we have the configuration of quartic curves having a double point and nine simple points. As a subcase is included a system of conics having a simple point in common and no basis points. This system is a linear combination of conics forming a net.

Type IV.

9. The defining equations for Type IV are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$, $v_i=0$ are cubic curves passing through six fixed points. We shall denote these points by P_i.

The equations of transformation are

$$\rho y_1=u_2v_3-u_3v_2=\phi_1(x), \text{ etc.}$$

The curves ϕ_i have a double point at each of the six P_i. They also intersect in $6\cdot6-6\cdot2\cdot2-3=9$ simple points of the system. We shall denote these basis points by Q_i. Each ϕ_i is of genus 4. The image in (y) of a line in (x) is a sextic curve.

The Fundamental System.

10. Each P_i of the net as well as each Q_i when substituted in the equations (1) and (2) make them identical. Hence they are fundamental points.

The image of each P_i in (y) is a conic which we shall denote by p_i. The residual image of $p_i(y)$ in (x), *i. e.*, r_i is a curve of order 12 having at the given P_i a five-fold point. The residual curve r_i has a double point at Q_i.

The image of each Q_i is a straight line q_i in (y) and a residual curve s_i in (x) of order 6 belonging to the net. At the given Q_i the curve r_i has a double point and a simple point at each of the remaining eight Q_i. The curve r_i has a double point at each P_i.

K is of order 15, genus 22; L is of order 12, and has thirty-three cusps. Aside from L the complete image of K is composed of nine straight lines taken twice, images of the nine basis points through which K passes twice and six conics taken five times, images of the six P_i through each of which K passes five times.

The complete image of L in (x) is a curve of order 72 of which K is a double component. The residual image J' is of order 42. The complete multiplicity of each P_i on K^2J' is 24 from which a five-fold point taken twice must be deducted for K^2. Hence P_i is of multiplicity 14 on J'. The total number of intersections at each Q_i is twelve, of which four are on K^2. J' has at each Q_i an eight-fold point. The genus of J' is 22. K and J' intersect in $42\cdot15-6\cdot14\cdot5-9\cdot2\cdot8=66$ points not at fundamental points which are thirty-three contacts corresponding to the thirty-three cusps on L.

Each q_i meets L in twelve points. As Type I, L is tangent to q_i as many times as K is multiple at Q_i. Hence there are two contacts of L with q_i and eight simple intersections each of which corresponds to a particular direction of J' through Q_i.

K intersects r_i in eight points not at fundamental points which correspond to the eight simple points of intersection of L with q_i.

J' has four simple intersections or two contacts with r_i which are images of the two contacts of L with q_i.

In the same way L meets p_i in twenty-four points of which five are contacts and fourteen are simple intersections. The points of tangency correspond to the contacts of K with s_i at P_i. Each of the simple intersections corresponds to a particular direction of J' through P_i.

There are fourteen intersections of K with s_i not at fundamental points, images of the fourteen intersections of L with p_i.

There are ten intersections of J' with s_i, not at fundamental points, which are five contacts corresponding to the five contacts of L with p_i.

Type V.

11. The defining equations for Type V are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves passing through nine fixed points, and $v_i=0$ represent curves of order 9, having a triple point at eight of the nine fixed points of $u_i=0$. The equations of transformation are

$$\rho y_1=u_2v_3=\phi_1(x), \quad \rho y_2=-u_1v_3=\phi_2(x), \quad \rho y_3=u_2v_1-u_1v_2=\phi_3(x).$$

There are eight points P_i of multiplicity four on each ϕ_i, and thirteen simple basis points Q_i. The fundamental curve $v_3=0$ has a three-fold point at each P_i and passes simply through twelve of the thirteen Q_i. $9\cdot 12-8\cdot 3\cdot 4=12$. We shall denote by Q_m the basis point through which $v_3=0$ does not pass.

The Fundamental System.

12. The image in (y) of each fundamental point Q_i is a straight line q_i passing through $P(y)$. The residual image in (x) is a composite curve of order 12 having eight points of multiplicity, four at $P(x)$, passing simply through eleven of the Q_i, and doubly through the given Q_i. One component

of this curve is $v_3=0$. The remaining image r_i is a curve of order 3 having a simple point at each P_i and also at the given Q_i and Q_m.

The image in (y) of Q_m is a straight line q_m which does not pass through $P(y)$. The residual image r_m in (x) is a curve of order 12 having a four-fold point at each P_i, a simple point at each Q_i, and a double point at Q_m.

The image of each P_i is a curve p_i of order 4 having a three-fold point at $P(y)$. The residual image in (x) is a composite curve of order 48, having a sixteen-fold point at seven P_i and a seventeen-fold point at the given P_i. The multiplicity at Q_i and Q_m is 4. One component is the image of $P(y)$ taken as many times as p_i passes through $P(y)$. The remaining component which is the curve s_i of order 21, having a multiplicity of 7 at seven of the P_i and of 8 at the given P_i. At each Q_i there is a simple point, but at Q_m there is a four-fold point.

The Jacobian is of order 33. This is a composite curve, one component being $v_3=0$. Hence the curve of coincidences K is of order 24, having eight-fold points at P_i, simple points at Q_i and a double point at Q_m. The genus of K is 28. The order of L is 18. Corresponding to each intersection of K with $v_3=0$, not at the fundamental points, is a branch of L through $P(y)$. $24 \cdot 9 - 8 \cdot 8 \cdot 3 - 12 \cdot 2 = 12$. Hence L has a twelve-fold point at $P(y)$. The genus of L is 28, and there are forty-two double points, or their equivalents, to be accounted for. The curve J' is of order 60.

The total multiplicity at P_i is 72, at Q_i the total multiplicity is 18, and at Q_m it is 18. Deducting the multiplicities at these points on K^2 and $v_3^{12}=0$ there remains a twenty-fold point at P_i, a four-fold point at Q_i, and a fourteen-fold at Q_m on J'. K and J' have, aside from the fundamental points, forty-two contacts which correspond to the forty-two cusps on L.

As in Type I, L is cut by q_m in eighteen points, two of which are contacts corresponding to the tangency of K with r_m at Q_m. The fourteen simple intersections of K with r_m, not at fundamental points, are images of the fourteen simple intersections of q_m with L.

The four points of intersection of J' with r_m, not at fundamental points, are two points of contact, images of the two contacts of L with r_m.

L has six points in common with q_i, not at $P(y)$, of which one is a point of contact and four are simple intersections. These correspond to the tangency of K with r_i at Q_i and to the directions of J' through Q_i.

The intersections of K with r_i, not at fundamental points are four, the images of the four intersections of L with q_i.

The intersections of J' with r_i, not at fundamental points, are two or one point of contact which is the image of the contact of L with q_i.

L has thirty-six intersections with p_i, not at $P(y)$. Eight of these are contacts and twenty are simple intersections. These correspond to the tangency of K with s_i at P_i and the multiplicity of J' at P_i.

There are twenty simple intersections of K with s_i, not at fundamental points, which are images of the simple intersections of L with p_i.

There are sixteen intersections of J' with s_i, not at fundamental points, which are eight contacts, images of eight contacts of L with p_i.

There are eight four-fold points and thirteen simple basis points in the general case of a system composed of a net of cubic curves, and a net of curves of order 9. A subcase of this is a linear system of curves of order 9 having eight triple points and six simple basis points. Hence the latter type which is used here is a subcase of the general system composed of the net of cubics and the net of curves of order 9.

Completeness of Enumeration.

13. The three images of a point in (y) may be defined as the variable intersections of systems of curves in (x) in many ways. They are, however, reducible to one of the five types described above. For example, the defining equations may be

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_2+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves having a double point at $P(x)$ and a simple point at $Q(x)$, and where $v_i=0$ represent conics passing through P and Q. $u_i=0$ have the form $x_3(x_1x_2)_2+x_1(x_1x_2)=0$, and the curves $v_i=0$ have the form $x_3(x_1x_2)+x_1(x_1x_2)=0$. In $\Sigma u_i=0$ there are ∞^5 degrees of freedom, and in $\Sigma v_i=0$ there are ∞^3 degrees of freedom.

If the $u_i=0$ have the added restriction that they pass simply through a third point $R=(1,0,0)$ through which $v_i=0$ do not pass, the net has ∞^2 degrees of freedom. We shall denote the net of these cubic curves through one double and two simple points by $y_1\psi_1+y_2\psi_2+y_3\psi_3=0$. By a quadratic inversion the ψ_i becomes curves of order 6 which contain the image of the double point P and the two simple points Q and R. The residual curve is of order 2 which passes once through P but not through R or Q.

The conic of the net $v_i=0$ through P and Q become composite curves of order 4 whose residual image is a conic passing through P and Q.

The curves of transformation $\rho y_i = \phi_i$ of the system $\Sigma \psi_i y_i = 0$, $\Sigma v_i y_i = 0$ obtained by finding the values of y_i are of order 5 having at P, and Q points of multiplicity 3 and 2 and of multiplicity 1 at the nine fixed basis points T_i. By a quadratic inversion the ϕ_i are of order 10. These are composite curves whose residual images are quartic curves, having one double point and nine simple points. The genus of the quintic is 2 as is also that of the quartic.

The curve of coincidences is of order 11 having at P, Q, and T_i multiplicities of 7, 4 and 2. By the inversion K is a composite curve of order 22 whose residual image is a curve of order 9 with a point of multiplicity 2 and 5 at P and T_i. The genus of K is 9.

Hence the ψ_i reduce to conics through one fixed point, v_i become conics through this same fixed point and a point M through which the first conics do not pass. The transformation ϕ_i reduce to quartics having a double point and nine simple points. K reduces to a curve of order 9 and of genus 9 with one five-fold point and nine double points. But this is exactly the configuration for Type III. Thus the system of a net of curves of order 3 having a fixed double point and a fixed simple point through which pass a net of conics reduces by a birational transformation to the system of Type III.

The same method of reduction can be employed in every case. The procedure is exactly that used by Bertini in the paper cited.

Cyclic Involutions.

14. In order that the discussion of the (1, 3) correspondences may be complete it is necessary to consider those cases in which the images of a point (y) are rationally separable such that to a point $P(y)$ correspond three points $P_1(x)$, $P_2(x)$, $P_3(x)$ having the following property. If P_1 describe a locus, P_2 describes another locus and P_3 another locus, all in (1, 1) correspondence. If P_1 and P_2 describe the same locus, P_3 describes the same locus. Hence a birational transformation exists by means of which the three points P_i are permuted among themselves. By this transformation $P_1 \backsim P_2$, $P_2 \backsim P_3$, $P_3 \backsim P_1$. Since the three image points form a group the transformation is cyclic and of order or period 3. Such an involution is called cyclic.

If two of the three image points coincide all three are identical. For the transformation which sends P_1 into P_2 also sends P_2 into P_3 and P_3 into P_1. Hence if $P_1 = P_2$ then $P_1 = P_2 = P_3$. Hence K and J' are identical. There is a birational relation between the curves K and L.

Bottari* has discussed the possible forms of all cyclic types by means of projection from configurations in space of three or more dimensions.

For a cyclic transformation of order 3 there are four types among which are included all the Cremona transformations of order 3.

(1) The Jonquières transformation of three points on a fixed line.

(2) The Jonquières transformation of three points on a pencil of lines p_i, the lines themselves being permuted by a linear transformation of order 3. In particular P_1 on p_1 is transformed into P_2 on p_2, P_2 on p_2 is transformed into P_3 on p_3 which in turn becomes the original point. In the same way P_2 and P_3 on p_1 have uniquely associated points on p_2 and p_3 such that the three points on each p_i form a distinct group.

(3) The three permuted points are the three intersections of two cubic curves through six basis points which form two triads of points in three-fold perspective. The transformation is accomplished by means of a quartic curve through six points. One triad consisting of double points, the other of simple points. Kantor† calls the type Δ_3.

(4) A particular case of Type V, in which the curves of order 9, and a pencil of cubic curves are invariant under a Cremona transformation of period 3 and order 13. This type is called N_3 by Kantor (*loc. cit.*, p. 288).

Cyclic Type I.

15. Given a curve C_n with a fixed point of multiplicity n–3. The three points in which a line through the fixed point cuts the curve are images of the point of intersection of two lines in (y). The points of each triad are on an invariant line.

The equations of the transformation have the form

$$\rho x_1' = \frac{ax_1+b}{cx_1+d}, \quad \rho x_2' = x_2, \quad \rho x_3' = x_3.$$

It is of period 3 when $ad+bc+a^2+d^2=0$. Equating T^2 and T^{-1} we have

$$x^2(acd+cd^2)+x(d^3-abc)-bcd-bd^2$$
$$=x^2(-a^2c-bc^2)+x(a^3-bcd)+a^2b+abd+abd.$$

The condition for this equality to exist is found by putting

$$b=\frac{-(a^2+d^2+ad)}{c}.$$

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*"Sulla razionalita dei piani multipli $\{x, y, \sqrt[n]{F(x,y)}\}$." *Annali di Matematica.* Series 3, Vol. II (1899), p. 278.

†"Premiers Fondements pour une Théorie des Transformations Periodiques Univoques," *Memoire, della R. Accademia.* Naples. Series 2, Vol. III, e Vol. IV, 2 (1891), pp. 1-335. See pp. 260-262.

Locus of Invariant Points.

16. The invariant points on each line descrive curves that are rationally separable, hence K consists of two distinct rational curves. The points will be invariant when $x_1'=x_1$. Thus; $x_1=\dfrac{ax_1+b}{cx_1+d}$. The two roots of the equation of x are

$$x_a=\frac{(a-d)+(a+d)\sqrt{-3}}{2c},\quad x_b=\frac{(a-d)-(a+d)\sqrt{-3}}{2c}.$$

A curve of order n having an n—3-fold point at $P(x)=(1,0,0)$ has the form

$$x_1^3u_i(x_{2_{n-3}}x_3)+x_1w_i(x_{2_{n-2}}x_3)+x_1w_i(x_{2_{n-1}}x_3)+s_i(x_2x_{3_n})=0.$$

A curve is invariant under T when its equation is of the form

$$(x_1-u_i)(x'_1-u_i)(x_1''-u_i)=k \tag{1}$$

for the curve ϕ_i of the net, wherein

$$x_1'=T(x),\quad x_1''=T^2(x).$$

If the curve (1) is cut by any line which passes through the fixed point T since the line is invariant under the transformation the three points are rationally separable and can be uniquely determined as a cyclic transformation of period 3.

Quadratic Cyclic Transformations.

17. The quadratic transformations of period n have been discussed by V. Snyder.* Of these transformations, those that concern us can be transformed into linear perspectivities.

Cyclic Type II. Non-perspective Jonquières.

18. In Type II the three images are the variable intersections of two curves ϕ_i and ψ_i of degrees n and m having at P points of multiplicity $n-3$ and $m-3$. Each of the three image points lies on a separate line of the pencil through P_1, the three lines being permuted by means of the transformation. The transformation has the form T in which a, b, c, d now have the previous relation and the additional restriction that x_2 appears only in powers of three in equation (1) of the curve above.

The curves ϕ_i and ψ_i are each invariant as a whole under the transformation. Hence, to an arbitrary point P_1 there correspond two other points P_2

* "Periodic Quadratic Transformations in the Plane," *Annals of Mathematics*, Series 2, Vol. XIII (1912), pp. 140–148.

and P_3 which are rationally distinct and lie on the curves. The k points occur in groups of three or $3k'=k$. The equations give us for y_i the values

$$\rho y_1=\phi_2\psi_3-\phi_3\psi_2\text{, etc.}$$

The images of lines in (y) are curves of order $m+n$, having at each k' point a multiplicity of $r+s$, where r is the multiplicity on ϕ_i and s the multiplicity on ψ_i. The curves $\phi_i=0$, $\psi_i=0$ are now to be restricted to pass through k' arbitrary points in the plane.

It is a curve $C_{k'}$ of order $r+k$ taken three times, and has a point of such multiplicity α_i as will make the curve rational. The complete image of this curve $C_{k'}$ in (x) is of order $(r+s)(m+n)-\Sigma(m+n)\alpha_i$.

The locus of K consists of the one line $x_3=0$.

Cyclic Type III.

19. Type III consists of the configuration of the three image points in (x) lying on a system of cubic curves with six basis points. In Type V of the general case to which this is related the defining equations are

$$u_1y_1+u_2y_2+u_3y_3=0, \quad v_1y_1+v_2y_2+v_3y_3=0,$$

where $u_i=0$, $v_i=0$ are cubic curves through six fixed points.

To build this type it is necessary to find a birational transformation for the points in (x) such that $T^3=1$.

A transformation composed of the product of two inversions will send a line into a quartic curve through three double points and three fixed simple points. The six basis points are disposed of as a triad of double points and a triad of simple points.

Let A, B, C; $\bar{A}$, $\bar{B}$, $\bar{C}$ be the two triads of fundamental points. We shall denote as double points $A=(1,0,0)$, $B=(0,1,0)$, $C=(0,0,1)$.

By the transformation T the line $A\bar{A}$ becomes a quartic curve passing through the image of A and of $\bar{A}$. The residual image of $A\bar{A}$ is a line $B\bar{B}$ through two of the remaining four fixed points. By a second transformation T^2 the line $B\bar{B}$ has for its residual image the line $C\bar{C}$. Since the transformation is of period 3 $C\bar{C}$ goes into $A\bar{A}$, and it has been shown by Kantor (*loc. cit.*, p. 260), that the two triads of points are in three-fold perspective. We shall denote as simple points $\bar{A}=(a,b,c)$, $\bar{B}=(a,\omega b,\omega^2 c)$, $\bar{C}=(a,\omega^2 b,\omega c)$.

This means that $A\bar{A}$ and the lines into which it is transformed by T and T^2 pass through a common point. Also $A\bar{C}$ and its associated lines pass through a common point. We shall denote these points by P, Q, and R.

In order to obtain the relations of the six points in (x) under the transformation, consider three planes x, x', x''. Under T the six (x') images of the six fundamental points (x) are obtained. These under T^2 define six points (x'') and these in turn go into the original (x) points. The (x') and (x'') planes are superposed on the (x) plane to obtain the required relations.

We shall consider the two triads of points in (x) and (x') as

$$(A_2B_2C_2\overline{A}_1\overline{B}_1\overline{C}_1) \quad \text{and} \quad (A_2'B_2'C_2'\overline{A}_1'\overline{B}_1'\overline{C}_1').$$

The points A_2, B_2, C_2 each have conics in (x') and the points $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_1$ have straight lines in (x') for their images. Also the point

$$\begin{array}{llll} A_2' \text{ goes into } & C_2(ABC\overline{B}\overline{C}), & \overline{A}_1' \text{ goes into } & C_1(BC), \\ B_2' \quad `` & C_2(ABC\overline{A}\overline{C}), & \overline{B}_1' \quad `` & C_1(AC), \\ C_2' \quad `` & C_2(ABC\overline{A}\overline{B}), & \overline{C}_1' \quad `` & C_1(AB). \end{array}$$

Since any straight line in (x') goes into a quartic through $(A_2B_2C_2\overline{A}_1\overline{B}_1\overline{C}_1)$, the image of the line $A'\overline{A}'$ is a quartic curve containing the image of A' and $\overline{A}'$. These are the straight line BC and the conic through $ABC\overline{B}\overline{C}$. The residual image is, then, the line $A\overline{A}$. In the same way the images of other lines may be obtained.

Since $CB \equiv x_1 = 0$, $AC \equiv x_2 = 0$, $AB \equiv x_3 = 0$, it follows that

$$\begin{aligned} x_1 &= k\,(bcx_1^2 - a^2x_2x_3)\,(abx_3^2 - c^2x_1x_2), \\ x_2 &= l\,(acx_2^2 - b^2x_1x_3)\,(bcx_1^2 - a^2x_2x_3), \\ x_3 &= m\,(acx_2^2 - b^2x_1x_3)\,(abx_3^2 - c^2x_1x_2). \end{aligned}$$

The points P, Q, and R are invariant under the transformation, hence the substitution of the coordinates of one of these points in the above equations gives the values of k, l, and m. These are ω^2ab, ωbc and ac, respectively.

The equations of transformation T^2 are

$$\begin{aligned} x_1' &= ab\,(bcx_1^2 - a^2x_2x_3)\,(abx_3^2 - c^2x_1x_2), \\ x_2' &= bc\,(acx_2^2 - b^2x_1x_3)\,(bcx_1^2 - a^2x_2x_3), \\ x_3' &= ac\,(acx_2^2 - b^2x_1x_3)\,(abx_3^2 - c^2x_1x_2). \end{aligned}$$

Similarly for T we obtain

$$x_1' = ax_1x_2\,(abx_3^2 - c^2x_1x_2), \quad x_2' = bx_2x_3\,(bcx_1^2 - a^2x_2x_3), \quad x_3' = cx_1x_3\,(acx_2^2 - b^2x_1x_3).$$

The nine points A, B, C, $\overline{A}$, $\overline{B}$, $\overline{C}$, P, Q, R lie on a cubic curve which is invariant, but since two triads of lines pass through these points, they are the basis points of a pencil of cubic curves. The equations have the form

$$A\overline{A} \cdot B\overline{C} \cdot \overline{B}C + \lambda C\overline{C} \cdot \overline{A}B \cdot A\overline{B}. \tag{1}$$

If any curve of the pencil is invariant under T, we must have

$$C\bar{C}\cdot\bar{A}B\cdot\bar{B}A+\bar{\lambda}B\bar{B}\cdot A\bar{C}\cdot C\bar{A}=0, \tag{2}$$

such that (1) and (2) define the same curve. Equation (1) has the form

$$(bx_3-cx_2)(\omega bx_1-ax_2)(\omega cx_1-ax_3)+\lambda(\omega^2bx_1-ax_2)(cx_1-ax_3)(\omega cx_2-bx_3)=0, \tag{3}$$

which by T goes into

$$(\omega^2bx_1-ax_2)(ax_3-cx_1)(bx_3-\omega cx_2)+\bar{\lambda}(ax_3-\omega^2cx_1)(\omega bx_3-cx_2)(bx_1-ax_2)=0. \tag{4}$$

By comparing the coefficients of like terms in equations (3) and (4) we obtain the relation $\bar{\lambda}=\frac{\omega}{\omega+\lambda}$. By the substitution of $\frac{\omega}{\omega+\lambda}$ for $\bar{\lambda}$ in equation (4) we find $\lambda=\omega$ and $\lambda=\omega^2$. The two curves are

$$bc^2x_1^2x_2+\omega ca^2x_2^2x_3+\omega^2ab^2x_1x_3^2=0, \tag{5}$$

$$b^2cx_1^2x_3+\omega^2ac^2x_1x_2^2+\omega a^2bx_2x_3^2=0. \tag{6}$$

The existence of the two cubic curves satisfied by different values of λ has been mentioned by H. S. White.*

It is necessary to determine whether either of these cubic curves contains points invariant under the transformation. The point of contact of a line through $A=(0,0,1)$ or $x_1=mx_2$ with (5) gives $x_1=\omega a\sqrt[3]{4}$, $x_2=b\sqrt[3]{4^2}$, $x_3=-2\omega c$. Under T these values become $x_1=(\sqrt[3]{4^2})\omega a$, $x_2=-2\omega b$, $x_3=\sqrt[3]{4}c$. Hence this point is not invariant. The point of contact of $x_1=mx_2$ with (6) gives $x_1=a\sqrt[3]{4^2}$, $x_2=b\omega^2\sqrt[3]{4}$, $x_3=-2c$. Under T this point is invariant, hence the curve (6) is the equation of $K=J'$.

All the cubic curves through $A, B, C, \bar{A}, \bar{B}, \bar{C}$ which contain the three permuted points which are images of a point (y) form a net of equianharmonic curves; they are tangent to one of the sides $x_1=0$, $x_2=0$, $x_3=0$ and do not pass through P, Q and R.

Since a line in (y) becomes a cubic curve in (x) any one of these equianharmonic curves may be taken for $\phi_i=0$ in the equations of transformation which written in the form of defining equations are $\rho y_i=\phi_i$.

The equation of L, the image of K, is also a cubic curve. To a line in (x) corresponds a cubic in (y) which has for its complete image in (x) a composite curve made up of the original line and two quartic curves. The line when operated upon by the transformation T becomes one of these quartic curves, and when operated upon by T^2 becomes the second quartic which accounts for the curves of order 9 as $C_1+C_4+C_4$.

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* "Plane Cubics and Irrational Covariant Cubics," *Transactions American Mathematical Society*, Vol. I (1900), pp. 170–178.

Non-perspective Linear Transformations.

20. It was shown in Art. 17 that quadratic transformations of period 3 can be reduced to the Jonquières type of cyclic involution.

We now consider conics through one fixed point, invariant under the non-perspective cyclic linear transformation of period 3.

If $\rho x_1'=\omega x_1$, $\rho x_2'=\omega^2 x_3$, $\rho x_3'=x_3\,(\omega^3=1)$, the expression $x_1x_2+kx_3^2$ remains invariant, $x_1x_3+lx_2^2$ is multiplied by ω^2, and $x_2x_3+mx_1^2$ is multiplied by ω by the transformation. Hence the systems of conics

$$x_1x_2y_1+x_3^2y_2=0, \quad x_2^2y_3+x_1x_3(y_1+y_2+y_3)=0$$

defines a triad of variable intersections which is invariant under the transformation.

For K we find $x_1x_2^2x_3^2(x_1x_2-x_3^2)(x_1x_3+x_2^2)=0$, which is satisfied only by fundamental curves. This is a particular case of Cyclic Type III.

Consider the ∞^3 system of cubic curves

$$\alpha_1x_1^3+\alpha_2x_2^3+\alpha_3x_3^3+6\alpha_4x_1x_2x_3=0.$$

Every curve goes into itself by the transformation. Any two curves of the system intersect in nine points, making three triads. Now choose any two points (x), $(\bar{x})$ as basis points of the system of cubic curves. Associated with (x) are two distinct points forming a triad, and associated with $(\bar{x})$ are two points so that the six points form two triads of permuted points. There remains one triad of variable points which are rationally distinct and are images of a point in (y).

Geometric Interpretation of Cyclic Type III.

21. The relation between any point of the (y) plane with its three corresponding images in the (x) plane has been developed for Type III by Bottari.*

Cyclic Type IV.

22. Type IV is defined by a system of curves of order 9 which are invariant under the transformation and on which lie the three variable points which are permuted among themselves.

The system has ∞^6 degrees of freedom as shown in the form

$$\lambda_1\psi_1+C_6(\lambda_2\phi_1+\lambda_3\phi_2)+\lambda_4\phi_1^3+\lambda_5\phi_1^2\phi_2+\lambda_6\phi_1\phi_2^2+\lambda_7\phi_2^3=0. \tag{1}$$

Since ψ, $\phi_i\phi_k$ are invariant under T, but C_6 is not, we may reject the terms involving C_6,

$$\lambda_1\psi+\lambda_2\phi_1^3+\lambda_3\phi_1^2\phi_2+\lambda_4\phi_1\phi_2^2+\lambda_3\phi_2^3=0.$$

* *Loc. cit.*, p. 282.

Two of the four constants in equation (2) may be chosen arbitrarily. We shall let one constant determine a point P on the ϕ_2 which also uniquely determines the two remaining associated points of the triad. Substituting the coordinates of this point in (2) there will be a fixed value $\lambda_1\psi_p$ and $\lambda_4\phi_p^3$ since P does not lie on ψ or ϕ_1. The resulting equation is then, $\lambda_1 k+\lambda_4 l=0$ or $\lambda_4=-\lambda_1\frac{k}{l}$. A second constant may be used to determine a point Q on ϕ_1 with which are also associated the two remaining points of the triad.

$$\lambda_1\psi+\phi_1^2\phi_2+\phi_1\phi_2^2-\lambda_1\frac{k_1}{l}\phi_1^3-\lambda_1\frac{k'}{l'}\phi_2^3=0,$$

or

$$\lambda_1(\psi+\phi_1^3+\phi_2^3)+\lambda_2\phi_1^2\phi_2+\lambda_3\phi_1\phi_2^2=0.$$

The curves $f_i(x)=0$ intersect into two variable triads.

The curves $f_i(x)$ intersect in six fixed points which are fundamental points. $9\cdot 9-8\cdot 3\cdot 3-3=6$ composed of two triads of points P_i and Q_i.

Through the point $(1, 0, 0)$ we have a pencil of lines of the form $\phi_1\phi_2(\phi_1 y_2+\phi_2 y_3)=0$. To the lines of the pencil correspond cubic curves in (x). The transformation curves $\rho y_i=f_i(x)$ with eight triple points and two triads of simple points have fundamental elements in (y) corresponding to their basis points. To P_i correspond lines p_i in (y) through $(1, 0, 0)$. The residual image of these lines is a curve of order 9 composed of a sextic, the image of the point, and a cubic, the image of the line. In the same way the residual image of Q_i is a cubic curve $\bar{Q}_i(x)$.

The triple points R_i on the $f_i(x)$ have for images rational curves of order 3 in (y) with a double point at $(1, 0, 0)$. The complete image in (x) is a composite curve r_i of order 27 containing the image of the fundamental point $(1, 0, 0)$ twice. The residual component $\bar{R}_i$ is of order 15, having eight points of multiplicity 5 at R_i.

The Jacobian of the system is a sextic having eight double points. It is of genus 2.

The curve L is also a sextic of genus 2. There are four branches of the curve through $(1, 0, 0)$ consisting of two sets of tangents which count for eight intersections. The common tangent for two branches can not cut the curve again, but an arbitrary line through $(1, 0, 0)$ cuts L in two other points corresponding to the two points of intersection of a cubic of the pencil with K. Two cubics do not meet K except at fundamental points.

L cuts p_i in two points not at $(1, 0, 0)$ and $\bar{P}_i$ has two intersections with K not at fundamental points. $6\cdot 3-8\cdot 1\cdot 2=2$. If p_i is tangent to L at

$(1, 0, 0)$, $\bar{P}_i$ has no simple intersection with K but one tangency. This is also true for q_i in (y) and $\bar{Q}_i$ in (x). L cuts r_i in ten points or five contacts. $6 \cdot 3 - 4 \cdot 2 = 10$. K is tangent to each branch of R at R_i which correspond to the tangency of L with r_i at the five points, and there are no simple intersections. $6 \cdot 15 - 8 \cdot 5 \cdot 2 - 2 \cdot 5 = 0$.

The Cyclic Transformation N_3.

23. The cyclic transformation of period 3 involving eight fundamental points is of order 13. By means of the transformation the image of a line is a curve of order 13 on which the eight fixed points are distributed as follows: one triad of four-fold points A, B, C; one triad of five-fold points $\bar{A}, \bar{B}, \bar{C}$; one triple point G, and one six-fold point $\bar{G}$. In order to obtain the relation of the eight P_i under the transformation we shall consider the two planes (x) and (x') on which are the points $A_4B_4C_4\bar{A}_5\bar{B}_5\bar{C}_5G_3\bar{G}_6 : A'_4B'_4C'_4\bar{A}'_5\bar{B}'_5\bar{C}'_5G'_3\bar{G}'_6$, respectively.

The image of A, B, and C are three rational quartic curves in (x') having three double points and five simple points, the image of $\bar{A}$, $\bar{B}$, and $\bar{C}$ are three rational curves of order 5 having six double points and two simple points; the image of G is a cubic curve with one double point and six simple points, and of $\bar{G}$ the image is a sextic curve having one triple point and seven double points.

The multiplicity of A, B, C on the Jacobian is 11, of $\bar{A}$, $\bar{B}$, $\bar{C}$ is 14, of G is 8 and of $\bar{G}$ is 17.

By the transformation the line $A'C'$ goes into the curve

$$C_{13}(A_4B_4C_4\bar{A}_5\bar{B}_5\bar{C}_5G_3\bar{G}_6).$$

The image of a point A' is the curve $C_4(A_1B_1C_1\bar{A}_1\bar{B}_2\bar{C}_2G_1\bar{G}_2)$ and of the point C' is the curve $C_4(A_1B_1C_1\bar{A}_2\bar{B}_2\bar{C}_1G_1\bar{G}_2)$. Hence the residual image of the line $A'C'$ is a quintic curve $C_5(A_2B_2C_2\bar{A}_2\bar{B}_1\bar{C}_2G_1\bar{G}_2)$.

In the same way the residual image of $A'B'$ is the curve

$$C_5(A_2B_2C_2\bar{A}_2\bar{B}_2\bar{C}_1G_1\bar{G}_2).$$

Consider the points (x') superposed upon the (x) points such that $A'B'C'\bar{A}'\bar{B}'\bar{C}'G'\bar{G}' \equiv \bar{B}\bar{C}\bar{A}BCA\bar{G}G$, respectively.

By T the point A' goes into

$$C_4(A_1B_1C_1\bar{A}_1\bar{B}_2\bar{C}_2G_1\bar{G}_2) \equiv C_4(\bar{C}'_1\bar{A}'_1\bar{B}'_1C'_1A'_2B'_2\bar{G}'_1G'_2).$$

By T^2 this curve becomes a composite curve of order $4 \cdot 13$ or

$$C_{52}(A_{16}B_{16}C_{16}\bar{A}_{20}\bar{B}_{20}\bar{C}_{20}G_{12}\bar{G}_{24}),$$

from which the components which are images of the points $\overline{C}'\overline{A}'\overline{B}'C'A'B'\overline{G}'G'$ are deducted as $C_{47}(A_{14}B_{14}C_{14}\overline{A}_{18}\overline{B}_{18}\overline{C}_{18}G_{10}\overline{G}_{23})$.

Hence the residual component is a curve

$$C_5(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_1G_2\overline{G}_1) \equiv C_5(\overline{C}'_2\overline{A}'_2\overline{B}'_2C'_2A'_2B'_1\overline{G}'_2G'_1).$$

By T again the quintic becomes a composite curve $C_{65}(A_{20}B_{20}C_{20}\overline{A}_{25}\overline{B}_{25}\overline{C}_{25}G_{15}\overline{G}_{30})$, from which the image of the fundamental points of the quintic are deducted as before.

Hence this curve passes through the point $A' = \overline{B}$ once more than the parameters provide for, so that this point which is the residual image of T^3 is the original point. The transformation is, then, of period 3.

In the same way it can be shown that the point $\overline{G}' \equiv G$ by T goes into $C_3(A_1B_1C_1\overline{A}_1\overline{B}_1\overline{C}_1G_0\overline{G}_2)$. By T^2 the residual image is $C_6(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_3\overline{G}_2)$, and by T^3 the image is composed of the curve $C_{78}(A_{24}B_{24}C_{24}\overline{A}_{30}\overline{B}_{30}\overline{C}_{30}G_{18}\overline{G}_{36})$ and the point G, so that the image of $\overline{G}' \equiv G$ is the same point.

Also the point $\overline{A}' = B$ goes into a curve $C_5(A_1B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_1\overline{G}_2)$ by T, and by T^2 the residual image is $C_4(A_2B_2C_1\overline{A}_1\overline{B}_1\overline{C}_1G_2\overline{G}_1)$. Under T^3 the image is the curve $C_{52}(A_{16}B_{16}C_{16}\overline{A}_{20}\overline{B}_{20}\overline{C}_{20}G_{12}\overline{G}_{24})$ and the point $\overline{A}' = B$, so that again T^3 is an identity.

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